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Determination of All Coherent Pairs

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A pair of quasi-definite linear functionals $\{u_0, u_1\}$ on the set of polynomials is called a coherent pair if their corresponding sequences of monic orthogonal polynomials $\{P_n\}$ and $\{T_n\}$ satisfy a relation

$$T_n = \frac{P'_{n+1}}{n+1} - \sigma_n \frac{P'_n}{n}, \qquad n \geqslant 1,$$

with σ_n non-zero constants. We prove that if $\{u_0, u_1\}$ is a coherent pair, then at least one of the functionals has to be classical, i.e. Hermite, Laguerre, Jacobi, or Bessel. A similar result is derived for symmetrically coherent pairs.

1. INTRODUCTION

Several authors studied polynomials orthogonal with respect to a Sobolev inner product of the form

$$\langle f, g \rangle_s = \int_a^b fg \, d\Psi_0 + \lambda \int_a^b f'g' \, d\Psi_1, \tag{1.1}$$

where Ψ_0 and Ψ_1 are distribution functions and $\lambda \geqslant 0$. For a survey of the theory we refer the reader to [5] and [11].

In [4] Iserles *et al.* introduced the concept of the coherent pair, which proved to be a very fruitful concept. It reads as follows. Let $\{P_n\}$ denote the monic orthogonal polynomial sequence (MOPS) with respect to $d\Psi_0$ and let $\{T_n\}$ denote the MOPS with respect to $d\Psi_1$, then $\{d\Psi_0, d\Psi_1\}$ is called a coherent pair if there exist non-zero constants σ_n such that

$$T_n = \frac{P'_{n+1}}{n+1} - \sigma_n \frac{P'_n}{n}, \qquad \text{for all} \quad n \geqslant 1. \tag{1.2}$$

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Iserles *et al.* showed that if $\{d\Psi_0, d\Psi_1\}$ is a coherent pair, then the sequence of polynomials $\{S_n^\lambda\}$ orthogonal with respect to the inner product (1.1) has an attractive structure. Put

$$S_n^\lambda = \sum_{m=1}^{n} \alpha_m^n(\lambda)\, P_m(x), \qquad n \geqslant 1,$$

then Iserles *et al.* showed that the normalization of P_n and S_n^λ can be changed by multiplying these functions with suitable constants in such a way that the coefficients α_m^n become independent of n, apart from the leading coefficient α_n^n. Write $\alpha_m = \alpha_m^n$ for $1 \leqslant m \leqslant n-1$, then α_m is a polynomial in λ of degree m and the polynomials $\alpha_m(\lambda)$ satisfy a three term recurrence relation. Moreover, if $\{d\Psi_0, d\Psi_1\}$ is a coherent pair, then the $\{S_n^\lambda\}$ satisfy a four term recurrence relation; see [2]. It is easy to prove that when $\{d\Psi_0, d\Psi_1\}$ is a coherent pair, λ is sufficiently large and $n \geqslant 2$, then S_n^λ has n different, real zeros interlacing with the zeros of P_{n-1} and with those of T_{n-1}; see [10]. Therefore it is interesting to investigate under what conditions $\{d\Psi_0, d\Psi_1\}$ is a coherent pair.

Marcellán and Petronilho [7] studied this problem in a more general setting where u_0 and u_1 are quasi-definite linear functionals on the space of polynomials and the corresponding MOPS satisfy a relation of the form (1.2). They solved the problem completely for the case when one of the functionals u_0, u_1 is a classical one, i.e. Hermite, Laguerre, Jacobi, or Bessel. In a recent paper [6], Marcellán, Pérez, and Piñar showed that if $\{u_0, u_1\}$ is a coherent pair of quasi-definite linear functionals, then both are semiclassical, i.e. there exist polynomials φ_i, ψ_i $(i=0,1)$ such that $D(\varphi_i u_i) = \psi_i u_i$ $(i=0,1)$, where D denotes the distributional differentiation. Moreover, they showed that there exist polynomials A and B, such that $Au_0 = Bu_1$ with degree $A \leqslant 3$, degree $B \leqslant 2$.

It is the aim of the present paper to solve the problem completely and to determine all coherent pairs $\{u_0, u_1\}$ of quasi-definite linear functionals. We will prove that at least one of the functionals u_0, u_1 has to be classical, so all coherent pairs of functionals $\{u_0, u_1\}$ are already determined in [7] (apart from some special cases which are not mentioned in [7]).

We show that there are only two cases:

(i) The functional u_0 is classical; there exist polynomials φ, ψ, ρ, degree $\varphi \leqslant 2$, degree $\psi =$ degree $\rho = 1$, such that $D(\varphi u_0) = \psi u_0$ and $\varphi u_0 = \rho u_1$.

(ii) The functional u_1 is classical; there exist polynomials φ, ψ, ρ, degree $\varphi \leqslant 2$, degree $\psi =$ degree $\rho = 1$, such that $D(\varphi u_1) = \psi u_1$ and $\varphi u_0 = \rho u_1$.

We remark that it is possible that both u_0 and u_1 are classical. In Section 2 we give the basic definitions, notations, and known results on functionals and coherent pairs of functionals. In Section 3 we show that every coherent pair $\{u_0, u_1\}$ belongs to case (i) or to case (ii). Moreover, we give all coherent pairs for linear functionals which can be represented by distribution functions.

In Section 4 the functionals are symmetric. A pair of symmetric functionals $\{u_0, u_1\}$ is called a symmetrically coherent pair if their corresponding MOPS $\{P_n\}$ and $\{T_n\}$ satisfy a relation

$$T_n = \frac{P'_{n+1}}{n+1} - \sigma_{n-1} \frac{P'_{n-1}}{n-1}, \qquad \text{for} \quad n \geqslant 2,$$

with σ_n non-zero constants. We prove that if $\{u_0, u_1\}$ is a symmetrically coherent pair, then at least one of the functionals has to be classical. Moreover, a division in two cases as for the coherent pairs is given.

2. BASIC DEFINITIONS AND RESULTS

Let u denote a linear functional defined on the space of polynomials $\mathscr{P}$. A sequence of monic polynomials $\{P_n\}$ is called a monic orthogonal polynomial sequence (MOPS) with respect to u if

(i) degree $P_n = n$, $n = 0, 1, 2, \ldots$,

(ii) $\langle u, P_n P_m \rangle = 0$, $n \neq m$, $n, m = 0, 1, 2, \ldots$,

(iii) $\langle u, P_n^2 \rangle = p_n \neq 0$, $n = 0, 1, 2, \ldots$.

There exists a MOPS with respect to u if and only if u is quasi-definite; see [3], Ch. I §3. In that case the MOPS is unique. In the sequel we always suppose the functionals to be quasi-definite.

The MOPS $\{P_n\}$ satisfies a three-term recurrence relation of the form (see [3], p. 18)

$$P_{n+1}(x) = (x - \beta_n)\, P_n(x) - \gamma_n P_{n-1}(x), \qquad n \geqslant 1,$$

with $\gamma_n \neq 0$ for $n \geqslant 1$, $P_0(x) \equiv 1$, $P_1(x) = x - \beta_0$.

If A is a polynomial and u a functional, then Au is defined by

$$\langle Au, p \rangle = \langle u, Ap \rangle, \qquad p \in \mathscr{P}.$$

If the polynomial A is not the zero-polynomial and degree $A = n$, then we can write

$$A = \sum_{k=0}^{n} c_k P_k,$$

with $c_n \neq 0$, so $\langle Au, P_n \rangle = c_n p_n \neq 0$. This implies that if A and B are polynomials with $Au = Bu$, then $\langle (A-B)\,u, p \rangle = 0$ for all $p \in \mathscr{P}$ and $A - B$ has to be the zero-polynomial, i.e. $A = B$.

The distributional derivative Du of the functional u is defined by

$$\langle Du, p \rangle = -\langle u, p' \rangle, \qquad p \in \mathscr{P}.$$

It is easy to check that we have for an arbitrary polynomial φ:

$$D(\varphi u) = \varphi' u + \varphi\, Du.$$

A functional u is called classical if it satisfies a relation

$$D(\varphi u) = \psi u,$$

with φ and ψ polynomials, degree $\varphi \leqslant 2$, degree $\psi = 1$.

The classical functionals and corresponding orthogonal polynomial sequences are the following ones, see [9], up to a linear transformation of the variable.

(i) degree $\varphi = 0$: Hermite polynomials $\{H_n\}$ with $\varphi(x) \equiv 1$, $\psi(x) = -2x$.

(ii) degree $\varphi = 1$: Laguerre polynomials $\{L_n^{(\alpha)}\}$ with $\alpha \notin \{-1, -2, ...\}$, $\varphi(x) = x$, $\psi(x) = -x + \alpha + 1$.

(iii) degree $\varphi = 2$ and φ has two different roots: Jacobi polynomials $\{P_n^{\alpha, \beta}\}$ with $\alpha, \beta, \alpha + \beta + 1 \notin \{-1, -2, ...\}$, $\varphi(x) = 1 - x^2$, $\psi(x) = -(\alpha + \beta + 2)\, x + \beta - \alpha$.

(iv) degree $\varphi = 2$ and φ has a double root: Bessel polynomials $\{B_n^{(\alpha)}\}$ with $\alpha \notin \{-2, -3, ...\}$, $\varphi(x) = x^2$, $\psi(x) = (\alpha + 2)\, x + 2$.

Finally we remark that for a quasi-definite functional u a relation $D(\varphi u) = cu$, with φ a non-zero polynomial and c a constant cannot be satisfied, since $c \neq 0$ would imply $\langle u, 1 \rangle = 0$ and $c = 0$ would imply $\langle \varphi u, p \rangle = 0$ for all $p \in \mathscr{P}$.

In the sequel we will use the following definition and notations: u_0 and u_1 denote quasi-definite linear functionals on $\mathscr{P}$, $\{P_n\}$ the MOPS with respect to u_0, $\{T_n\}$ the MOPS with respect to u_1,

$$\langle u_0, P_n^2 \rangle = p_n \neq 0, \qquad n = 0, 1, 2, ...,$$

$$\langle u_1, T_n^2 \rangle = t_n \neq 0, \qquad n = 0, 1, 2, ...$$

The pair $\{u_0, u_1\}$ is called a coherent pair if there exist non-zero constants σ_n such that

$$T_n = \frac{P'_{n+1}}{n+1} - \sigma_n \frac{P'_n}{n} \qquad \text{for} \quad n \geqslant 1. \tag{2.1}$$

For a coherent pair we introduce the polynomials

$$C_n = \sigma_n \frac{T_n}{t_n} - \frac{T_{n-1}}{t_{n-1}}, \qquad n = 1, 2, \dots. \tag{2.2}$$

Then the leading coefficient of C_n is $\sigma_n / t_n \neq 0$.

The following basic proposition is due to Marcellán, Pérez, Piñar [6].

PROPOSITION 1. *Let $\{u_0, u_1\}$ denote a coherent pair, then*

$$n \frac{P_n}{p_n} u_0 = D(C_n u_1) \qquad \textit{for} \quad n \geqslant 1.$$

COROLLARY 1. *Let $\{u_0, u_1\}$ denote a coherent pair. Then*

$$\varphi\, Du_1 = \pi u_1, \qquad \varphi u_0 = B u_1, \qquad \pi u_0 = B\, Du_1,$$

with

$$\varphi = 2 \frac{P_2}{p_2} C_1 - \frac{P_1}{p_1} C_2, \tag{2.3}$$

$$\pi = -2 \frac{P_2}{p_2} C'_1 + \frac{P_1}{p_1} C'_2, \tag{2.4}$$

$$B = C_1 C'_2 - C'_1 C_2, \tag{2.5}$$

where degree $\varphi \leqslant 3$, degree $\pi \leqslant 2$, degree $B = 2$.

Proof. Proposition 1 with $n = 1$ and $n = 2$ reads:

$$\frac{P_1}{p_1} u_0 = C'_1 u_1 + C_1\, Du_1, \tag{2.6}$$

$$2 \frac{P_2}{p_2} u_0 = C'_2 u_1 + C_2\, Du_1. \tag{2.7}$$

Elimination of u_0 gives the first result, elimination of Du_1 the second one and elimination of u_1 the last one. The coefficient of x^n in the polynomial

C_n defined by (2.2) is σ_n/t_n; thus the coefficient of x^2 in the polynomial B defined in (2.5) is $\sigma_1\sigma_2/t_1t_2 \neq 0$; then B has degree 2. ∎

3. DETERMINATION OF COHERENT PAIRS

In this section we suppose that $\{u_0, u_1\}$ is a coherent pair and we use the notations of Section 2. We will prove that at least one of the functionals u_0, u_1 is classical. The polynomial B defined in (2.5) is of degree 2 and therefore has two zeros ξ_1 and ξ_2. We will prove that if $\xi_1 = \xi_2$, then u_0 is classical (Theorem 1) and if $\xi_1 \neq \xi_2$, then u_1 has to be classical (Theorem 2).

If the polynomial B in (2.5) has a double zero, then the situation is simple.

THEOREM 1. *Let $\{u_0, u_1\}$ denote a coherent pair of quasi-definite linear functionals. Suppose that the polynomial B in (2.5) has a double zero ξ. Then*

(i) *u_0 is classical with $D(\tilde{\varphi}u_0) = \psi u_0$ for some polynomials $\tilde{\varphi}$, ψ, degree $\tilde{\varphi} \leqslant 2$, degree $\psi = 1$;*

(ii) *$\tilde{\varphi}u_0 = (\sigma_1\sigma_2/t_1t_2)(x-\xi)\,u_1$.*

Proof. From (2.5) we obtain

$$0 = B'(\xi) = C_1(\xi)\,C_2''(\xi).$$

Hence $C_1(\xi) = 0$. Then applying again (2.5) we have $0 = B(\xi) = -C_1'(\xi)\,C_2(\xi)$, so $C_2(\xi) = 0$. Then (2.3) implies $\varphi(\xi) = 0$. Write $\varphi(x) = (x-\xi)\,\tilde{\varphi}(x)$.

Since $C_1(\xi) = C_2(\xi) = 0$, the polynomial C_1 divides C_2. Then the elimination of Du_1 from (2.6) and (2.7) can be done in such a way, that one arrives at

$$\tilde{\varphi}u_0 = \frac{\sigma_1\sigma_2}{t_1t_2}(x-\xi)\,u_1.$$

Then using (2.6),

$$D(\tilde{\varphi}u_0) = \frac{\sigma_1\sigma_2}{t_1t_2}D((x-\xi)\,u_1) = \frac{\sigma_2}{t_2}D(C_1u_1) = \frac{\sigma_2}{t_2}\frac{P_1}{p_1}u_0 = \psi u_0,$$

where ψ is a polynomial of degree 1, i.e. u_0 is classical. ∎

If B in (2.5) has two different zeros, the analysis is more complicated. We first derive some auxiliary results.

It follows from Proposition 1 and Corollary 1, that for $n \geqslant 1$,

$$n\frac{P_n}{p_n}Bu_1 = n\frac{P_n}{p_n}\varphi u_0 = \varphi D(C_n u_1)$$
$$= \varphi C'_n u_1 + \varphi C_n\, Du_1 = (\varphi C'_n + C_n \pi)\, u_1.$$

Hence

$$n\frac{P_n}{p_n}B = C'_n\varphi + C_n\pi, \qquad n \geqslant 1. \tag{3.1}$$

Lemma 1. *Suppose that ξ is such that $B(\xi)=0$, $\varphi(\xi)\neq 0$. Then there exists a k, independent of n, $k\neq 0$, such that*

$$C_n(\xi) + kC'_n(\xi) = 0 \qquad \textit{for all} \quad n \geqslant 1.$$

Proof. Substitution of ξ in (3.1) gives

$$C'_n(\xi)\,\varphi(\xi) + C_n(\xi)\,\pi(\xi) = 0, \qquad n \geqslant 1.$$

Consider the relation for $n=1$. Then $C'_1 = \sigma_1/t_1 \neq 0$ and $\varphi(\xi)\neq 0$ imply $\pi(\xi)\neq 0$.

Hence

$$C_n(\xi) + kC'_n(\xi) = 0 \qquad \text{for all} \quad n \geqslant 1,$$

with

$$k = \frac{\varphi(\xi)}{\pi(\xi)} \neq 0. \quad \blacksquare$$

Lemma 2. *Suppose that there exist ξ_1, ξ_2, $k_1 \neq 0$, $k_2 \neq 0$ such that*

$$C_n(\xi_1) + k_1 C'_n(\xi_1) = 0 \qquad \textit{and} \qquad C_n(\xi_2) + k_2 C'_n(\xi_2) = 0,$$

for all $n \geqslant 1$. Then $\xi_1 = \xi_2$ and $k_1 = k_2$.

Proof. Using the definition of C_n in (2.2) we obtain for ξ_j, ($j = 1, 2$),

$$\sigma_n\left\{\frac{T_n(\xi_j)}{t_n} + k_j\frac{T'_n(\xi_j)}{t_n}\right\} = \frac{T_{n-1}(\xi_j)}{t_{n-1}} + k_j\frac{T'_{n-1}(\xi_j)}{t_{n-1}}.$$

Put

$$h_n^{(j)}(\xi_j)=\frac{T_n(\xi_j)}{t_n}+k_j\frac{T'_n(\xi_j)}{t_n},\qquad n\geqslant 0,\quad j=1,2, \tag{3.2}$$

then

$$\sigma_n h_n^{(j)}(\xi_j)=h_{n-1}^{(j)}(\xi_j),\qquad n\geqslant 1,\quad j=1,2. \tag{3.3}$$

Note that

$$h_0^{(j)}(\xi_j)=\frac{1}{t_0}\neq 0,$$

and (3.3) implies $h_n^{(j)}(\xi_j)\neq 0$ for all $n\geqslant 0$. Dividing the relations (3.3) for $j=1$ and $j=2$ we obtain

$$\frac{h_n^{(1)}(\xi_1)}{h_n^{(2)}(\xi_2)}=\frac{h_{n-1}^{(1)}(\xi_1)}{h_{n-1}^{(2)}(\xi_2)},\qquad n\geqslant 1,$$

and by repeated application

$$\frac{h_n^{(1)}(\xi_1)}{h_n^{(2)}(\xi_2)}=\frac{h_0^{(1)}(\xi_1)}{h_0^{(2)}(\xi_2)}=1,$$

or

$$h_n^{(1)}(\xi_1)=h_n^{(2)}(\xi_2)\qquad \text{for all}\quad n\geqslant 0.$$

But now (3.2) gives

$$T_n(\xi_1)+k_1T'_n(\xi_1)=T_n(\xi_2)+k_2T'_n(\xi_2)\qquad \text{for all}\quad n\geqslant 0. \tag{3.4}$$

It follows that every polynomial p satisfies

$$p(\xi_1)+k_1p'(\xi_1)=p(\xi_2)+k_2p'(\xi_2).$$

Choose $p(x)=(x-\xi_1)^n$ then

$$(\xi_2-\xi_1)^n+nk_2(\xi_2-\xi_1)^{n-1}=0,\qquad n\geqslant 2,$$

and, as a consequence, $\xi_1=\xi_2$. Finally $k_1=k_2$. ∎

LEMMA 3. *Let B, φ, and π denote the polynomials defined in Corollary 2. Suppose that B has two different zeros. Then at least one of them is also a zero of φ. If $B(\xi)=\varphi(\xi)=0$, then $C_1(\xi)\neq 0$ and $\pi(\xi)=0$.*

Proof. It is a direct consequence of Lemma 1 and Lemma 2, that B and φ have at least one zero ξ in common. Since ξ is a simple zero of B, it follows $B'(\xi)\neq 0$. By using (2.5) $B'=C_1 C_2''$, hence $C_1(\xi)\neq 0$. Substituting ξ in (3.1) with $n=1$, we obtain $\pi(\xi)=0$. ∎

We now are able to treat the situation that B in (2.5) has two different zeros.

THEOREM 2. *Let $\{u_0, u_1\}$ denote a coherent pair of quasi-definite linear functionals. Suppose that the polynomial B in* (2.5) *has two different zeros. Then*

(i) *u_1 is classical with $D(\tilde{\varphi}u_1)=\psi u_1$ for some polynomials $\tilde{\varphi}$, ψ, degree $\tilde{\varphi}\leqslant 2$, degree $\psi=1$;*

(ii) *there exists a ξ such that*

$$\tilde{\varphi}u_0=\frac{\sigma_1\sigma_2}{t_1 t_2}(x-\xi)\,u_1.$$

Proof. Let ξ_1, ξ_2 denote the different zeros of B. By Lemma 3 at least one of them is also a zero of φ. Without loss of generality we may suppose $\varphi(\xi_1)=0$. Then by Lemma 3 also $\pi(\xi_1)=0$.

Put $B=(x-\xi_1)\,\tilde{B}$, i.e. $\tilde{B}=(\sigma_1\sigma_2/t_1 t_2)(x-\xi_2)$, $\varphi=(x-\xi_1)\,\tilde{\varphi}$, $\pi=(x-\xi_1)\,\pi_1$.

Then (3.1) reduces to

$$n\frac{P_n}{p_n}\tilde{B}=C_n'\tilde{\varphi}+C_n\pi_1,\qquad n\geqslant 1. \tag{3.5}$$

Moreover, the relations $\varphi u_0=Bu_1$, $\pi u_0=B\,Du_1$ and $\varphi\,Du_1=\pi u_1$ from Corollary 1 reduce to

$$\tilde{\varphi}u_0=\tilde{B}u_1+M\delta(\xi_1), \tag{3.6}$$

$$\tilde{B}\,Du_1=\pi_1 u_0+N\delta(\xi_1), \tag{3.7}$$

$$\tilde{\varphi}\,Du_1=\pi_1 u_1+K\delta(\xi_1), \tag{3.8}$$

for some constants M, N and K.

From (3.5) and Proposition 1 we obtain for $n \geqslant 1$,

$$(C'_n \tilde{\varphi} + C_n \pi_1)\, u_0 = n \frac{P_n}{p_n} \tilde{B} u_0 = \tilde{B}(C'_n u_1 + C_n\, Du_1),$$

or

$$C'_n(\tilde{\varphi} u_0 - \tilde{B} u_1) = C_n(\tilde{B}\, Du_1 - \pi_1 u_0),$$

and with (3.6) and (3.7)

$$MC'_n(\xi_1) = NC_n(\xi_1), \qquad n \geqslant 1. \tag{3.9}$$

Observe that $C'_1(\xi_1) \neq 0$ and, by Lemma 3, $C_1(\xi_1) \neq 0$; so $M = 0$ if and only if $N = 0$.

For the second zero ξ_2 of B there are two possibilities: $\varphi(\xi_2) \neq 0$ or $\varphi(\xi_2) = 0$.

(i) Let $\varphi(\xi_2) \neq 0$. Then Lemma 1 implies that there exists a non-zero constant k such that

$$C_n(\xi_2) + kC'_n(\xi_2) = 0 \qquad \text{for all} \quad n \geqslant 1.$$

Since $\xi_1 \neq \xi_2$ we conclude from Lemma 2 that (3.9) only can be satisfied with $M = N = 0$.

(ii) Let $\varphi(\xi_2) = 0$. Then we may proceed with ξ_2 as with ξ_1 and conclude that there exist constants M_2 and N_2, such that

$$M_2 C'_n(\xi_2) = N_2 C_n(\xi_2) \qquad \text{for all} \quad n \geqslant 1, \tag{3.10}$$

where $C'_1(\xi_2) \neq 0$, $C_1(\xi_2) \neq 0$.

Again Lemma 2 implies that at least one of the relations (3.9) and (3.10) has to be a trivial one. Without loss of generality we may suppose that (3.9) is trivial, i.e $M = N = 0$.

In both cases (3.6) reduces to

$$\tilde{\varphi} u_0 = \tilde{B} u_1 = \frac{\sigma_1 \sigma_2}{t_1 t_2} (x - \xi_2)\, u_1 \tag{3.11}$$

This proves assertion (ii) of the theorem.

To prove the first assertion we use (2.6) and (3.5) with $n=1$:

$$\tilde{\varphi}\,\frac{P_1}{p_1}\,u_0=\tilde{\varphi}C_1'u_1+\tilde{\varphi}C_1\,Du_1=\left(\frac{P_1}{p_1}\,\tilde{B}-C_1\pi_1\right)u_1+\tilde{\varphi}C_1\,Du_1,$$

or

$$\frac{P_1}{p_1}\,(\tilde{\varphi}u_0-\tilde{B}u_1)=C_1(\tilde{\varphi}\,Du_1-\pi_1u_1).$$

With (3.11) and (3.8) we obtain $KC_1(\xi_1)=0$. Since, by Lemma 3, $C_1(\xi_1)\neq 0$, we have $K=0$ and (3.8) reduces to

$$\tilde{\varphi}\,Du_1=\pi_1u_1.$$

Finally $D(\tilde{\varphi}u_1)=\tilde{\varphi}'u_1+\tilde{\varphi}\,Du_1=(\tilde{\varphi}'+\pi_1)\,u_1=\psi u_1$, where ψ is a polynomial of degree $\leqslant 1$. Since u_1 is quasi-definite the degree of ψ has to be 1; thus u_1 is classical. ∎

EXAMPLES. A linear functional is positive-definite if and only if it can be represented by a distribution function Ψ as (see [3], Ch. II)

$$\langle u,p\rangle=\int_a^b p(x)\,d\Psi(x),\qquad p\in\mathscr{P}.$$

Then a coherent pair of positive-definite linear functionals $\{u_0,u_1\}$ corresponds to a coherent pair of distribution functions $\{d\Psi_0,d\Psi_1\}$. We mention all coherent pairs of distribution functions which follow from Theorem 1 and 2. The classical polynomials are given in their usual notation (see e.g. Szegö [12]) and not in their monic version; a linear change in the variable gives again a coherent pair.

A. *Laguerre Case.* The distribution function $d\Psi(x)=x^\alpha e^{-x}\,dx$ with $\alpha>-1$ on $(0,\infty)$ defines a positive-definite classical functional u. The functional u satisfies $D(\varphi u)=\psi u$ with $\varphi(x)=x$.

From Theorem 1 and Theorem 2 we obtain the following coherent pairs.

$$d\Psi_0(x)=x^\alpha e^{-x}\,dx,\qquad d\Psi_1(x)=\frac{1}{x-\xi}\,x^{\alpha+1}e^{-x}\,dx+M\delta(\xi),\tag{3.12}$$

where we have to take $\alpha>-1$, $\xi\leqslant 0$, $M\geqslant 0$.

$$d\Psi_0(x)=(x-\xi)\,x^{\alpha-1}e^{-x}\,dx,\qquad d\Psi_1(x)=x^\alpha e^{-x}\,dx,\tag{3.13}$$

where $\xi < 0$, $\alpha > 0$.

$$d\Psi_0(x) = e^{-x}\,dx + M\delta(0), \qquad d\Psi_1(x) = e^{-x}\,dx, \tag{3.14}$$

with $M \geqslant 0$. In (3.12) the $d\Psi_1$ has to be interpreted as

$$\int_{-\infty}^{\infty} f(x)\,d\Psi_1(x) = \int_0^{\infty} f(x)\,\frac{1}{x-\xi}\,x^{\alpha+1}e^{-x}\,dx + Mf(\xi),$$

so the spectrum of Ψ_1 is $[0, \infty) \cup \{\xi\}$. The spectrum of all other distribution functions is $[0, \infty)$. It is not difficult to check that (3.12), (3.13) and (3.14) indeed define coherent pairs. For (3.12) and (3.13) compare [7]. Since (3.14) has not been mentioned in [7] we give a proof of it.

Let $\{P_n\}$ denote an orthogonal polynomial sequence with respect to $d\Psi_0$. Since $L_n^{(0)}(0) = 1$ for all $n \geqslant 0$ (see [12], 5.1.7) we have

$$\int_{-\infty}^{\infty} \{L_n^{(0)} - L_{n-1}^{(0)}\}\,P_k\,d\Psi_0 = \int_0^{\infty} \{L_n^{(0)} - L_{n-1}^{(0)}\}\,P_k e^{-x}\,dx = 0$$

if $k \leqslant n-2$. This implies

$$L_n^{(0)} - L_{n-1}^{(0)} = c_n P_n + c_{n-1} P_{n-1},$$

for some constants c_n and c_{n-1}. Then differentiation gives (compare [12], p. 102)

$$L_{n-1}^{(0)} = -c_n P_n' - c_{n-1} P_{n-1}'.$$

Remark. If $\alpha \neq 0$, then (3.7) and (3.8) with $N = K = 0$ imply that $d\Psi_0$ in (3.13) cannot have a term $M\delta(0)$.

B. *Jacobi Case.* The distribution function $d\Psi(x) = (1-x)^{\alpha}\,(1+x)^{\beta}$ with $\alpha > -1$, $\beta > -1$ on $(-1, 1)$ represents a positive-definite classical functional u with $D(\varphi u) = \psi u$, where $\varphi(x) = 1 - x^2$.

Theorem 1 and Theorem 2 give the coherent pairs

$$\begin{aligned} d\Psi_0(x) &= (1-x)^{\alpha}\,(1+x)^{\beta}\,dx, \\ d\Psi_1(x) &= \frac{1}{|x-\xi|}\,(1-x)^{\alpha+1}\,(1+x)^{\beta+1}\,dx + M\delta(\xi), \end{aligned} \tag{3.15}$$

with $\alpha > -1$, $\beta > -1$, $|\xi| \geqslant 1$, $M \geqslant 0$,

$$d\Psi_0(x) = |x-\xi|\,(1-x)^{\alpha-1}\,(1-x)^{\beta-1}\,dx, \qquad d\Psi_1(x) = (1-x)^{\alpha}\,(1+x)^{\beta}\,dx \tag{3.16}$$

with $|\xi| > 1$, $\alpha > 0$, $\beta > 0$,

$$d\Psi_0(x) = (1+x)^{\beta-1}\,dx + M\delta(1), \qquad d\Psi_1(x) = (1+x)^{\beta}\,dx, \tag{3.17}$$

with $\beta > 0$, $M \geqslant 0$ and

$$d\Psi_0(x) = (1-x)^{\alpha-1}\,dx + M\delta(-1), \qquad d\Psi_1(x) = (1-x)^{\alpha}\,dx, \tag{3.18}$$

with $\alpha > 0$, $M \geqslant 0$.

The spectrum of Ψ_1 in (3.15) is $[-1, 1] \cup \{\xi\}$; the spectrum of the other distribution functions is $[-1, 1]$.

Again it is easy to check that this indeed are coherent pairs (for (3.15) and (3.16) compare [7]). The coherence of (3.17) follows with $P_n^{(0,\beta-1)}(1) = 1$ for all $n \geqslant 0$ (see [12], (4.1.1)) and

$$\frac{d}{dx}\,(P_n^{(0,\beta-1)} - P_{n-1}^{(0,\beta-1)}) = \frac{1}{2}\,(2n+\beta-1)\,P_{n-1}^{(0,\beta)},$$

(see [1], p. 782). The coherence of (3.18) follows in a similar way.

C. *Hermite Case.* In the Hermite case the distribution function is $d\Psi(x) = e^{-x^2}\,dx$ on $(-\infty, \infty)$ with $\varphi(x) \equiv 1$. Theorem 1 and 2 imply that there cannot exist coherent pairs.

4. SYMMETRICALLY COHERENT PAIRS

In this section u_0 and u_1 denote *symmetric* quasi-definite linear functionals and $\{P_n\}$ and $\{T_n\}$ the corresponding MOPS. The polynomials of even degree are even functions and the polynomials of odd degree odd ones. In this situation (2.1) only can be satisfied with $\sigma_n = 0$ for all $n \geqslant 1$. Therefore Iserles *et al.* [4] introduced the concept of symmetrically coherent pair. The pair $\{u_0, u_1\}$ of symmetric functionals is called a symmetrically coherent pair if there exist non-zero constants σ_n such that

$$T_n = \frac{P'_{n+1}}{n+1} - \sigma_{n-1}\,\frac{P'_{n-1}}{n-1} \qquad \text{for} \quad n \geqslant 2.$$

In this section we assume $\{u_0, u_1\}$ to be a symmetrically coherent pair and we will prove that again at least one of the functionals has to be classical. Therefore we will use the polynomials

$$C_n = \sigma_{n-1} \frac{T_n}{t_n} - \frac{T_{n-2}}{t_{n-2}}, \qquad n \geqslant 1.$$

Proposition 1 is replaced by Proposition 2 which can be proved in the same way.

PROPOSITION 2. *Let $\{u_0, u_1\}$ denote a symmetrically coherent pair, then*

$$n \frac{P_n}{p_n} u_0 = D(C_{n+1} u_1) \qquad \textit{for} \quad n \geqslant 1.$$

COROLLARY 2. *Let $\{u_0, u_1\}$ denote a symmetrically coherent pair, then*

$$\varphi \, Du_1 = \pi u_1, \qquad x\varphi u_0 = xBu_1, \qquad \pi u_0 = B \, Du_1$$

with

$$\varphi = 3 \frac{P_3}{xp_3} C_2 - \frac{P_1}{xp_1} C_4, \tag{4.1}$$

$$\pi = -3 \frac{P_3}{xp_3} C_2' + \frac{P_1}{xp_1} C_4', \tag{4.2}$$

$$B = \frac{1}{x} \{C_2 C_4' - C_4 C_2'\}, \tag{4.3}$$

where degree $\varphi \leqslant 4$, degree $\pi \leqslant 3$ and degree $B = 4$.

Proof. Proposition 2 with $n = 1$ and $n = 3$ reads

$$\frac{P_1}{p_1} u_0 = C_2' u_1 + C_2 \, Du_1, \tag{4.4}$$

$$3 \frac{P_3}{p_3} u_0 = C_4' u_1 + C_4 \, Du_1, \tag{4.5}$$

where P_1, P_3, C_2' and C_4' are odd polynomials. Elimination of u_0 gives the first identity of Corollary 2. Elimination of Du_1 gives the second and elimination of u_1 gives the last relation. The leading coefficient of B is $2(\sigma_1 \sigma_3 / t_2 t_4) \neq 0$. ▮

All above mentioned polynomials are either even or odd. Then all zeros, apart from $x=0$ in the odd polynomials, appear in pairs $\{-\xi, \xi\}$. A result similar to Corollary 2 has been given in [8] based on Proposition 2 with $n=1$ and $n=2$. We have chosen the definition of B in such a way that we have the next lemma.

LEMMA 4. (i) *If B in (4.3) is of the form $B=2(\sigma_1\sigma_3/t_2t_4)(x^2-\xi^2)^2$, then $C_2=(\sigma_1/t_2)(x^2-\xi^2)$ and $(x^2-\xi^2) \mid C_4$.*

(ii) *If $C_2 \mid B$, then B is of the form $B=(2\sigma_1\sigma_3/t_2t_4)(x^2-\xi^2)^2$.*

Proof. Put $C_2=(\sigma_1/t_2)(x^2-\alpha^2)$ and $C_4=(\sigma_3/t_4)(x^4+\beta^2x^2+\gamma^2)$. Then (4.3) gives

$$B=\frac{2\sigma_1\sigma_3}{t_2t_4}(x^4-2\alpha^2x^2-\alpha^2\beta^2-\gamma^2).$$

(i) If $B=2(\sigma_1\sigma_3/t_2t_4)(x^2-\xi^2)^2$, then $\alpha^2=\xi^2$ and $-\alpha^2\beta^2-\gamma^2=\xi^4$. This implies $C_2=(\sigma_1/t_2)(x^2-\xi^2)$ and $C_4(\xi)=0$, i.e. $(x^2-\xi^2) \mid C_4$.

(ii) If $C_2 \mid B$, then $B(\alpha)=0$, i.e. $-\alpha^4-\alpha^2\beta^2-\gamma^2=0$ and $B=(2\sigma_1\sigma_3/t_2t_4)(x^2-2\alpha^2x^2+\alpha^4)$ has the desired form. ∎

Lemma 4 enables us to characterize $\{u_0, u_1\}$ in the case that B is a pure square.

THEOREM 3. *Let $\{u_0, u_1\}$ denote a symmetrically coherent pair of quasi-definite linear functionals. Let B in (4.3) be of the form $B=(2\sigma_1\sigma_3/t_2t_4)(x^2-\xi^2)^2$. Then*

(i) *u_0 is classical with $D(\tilde{\varphi}u_0)=\psi u_0$ for some polynomials $\tilde{\varphi}$, ψ, degree $\tilde{\varphi} \leqslant 2$, degree $\psi=1$;*

(ii) $\tilde{\varphi}u_0=2(\sigma_1\sigma_3/t_2t_4)(x^2-\xi^2)\,u_1$.

Proof. It follows from Lemma 4(i) and (4.1) that we can write $\varphi=(x^2-\xi^2)\,\tilde{\varphi}$, for a polynomial $\tilde{\varphi}$ with degree $\tilde{\varphi} \leqslant 2$. The elimination of Du_1 from (4.4) and (4.5) can be done in such a way that we obtain

$$x\tilde{\varphi}u_0=x\,\frac{2\sigma_1\sigma_3}{t_2t_4}(x^2-\xi^2)\,u_1, \tag{4.6}$$

i.e.

$$\tilde{\varphi}u_0=2\,\frac{\sigma_1\sigma_3}{t_2t_4}(x^2-\xi^2)\,u_1+M\delta(0), \tag{4.7}$$

for some constant M. We will show that $M=0$. Then u_0 is classical, since by (4.4),

$$D(\tilde{\varphi}u_0)=2\frac{\sigma_3}{t_4}D(C_2u_1)=2\frac{\sigma_3}{t_4}\frac{P_1}{p_1}u_0.$$

In order to prove that $M=0$ in (4.7) we use Proposition 2 with $n=2$:

$$2\frac{P_2}{p_2}u_0=C_3'u_1+C_3\,Du_1. \tag{4.8}$$

Elimination of Du_1 from (4.4) and (4.8) gives

$$\left(2\frac{P_2}{p_2}C_2-\frac{P_1}{p_1}C_3\right)u_0=(C_2C_3'-C_3C_2')\,u_1,$$

which will be abbreviated as

$$q_4u_0=b_4u_1, \tag{4.9}$$

where q_4 and b_4 are even polynomials, degree $q_4\leqslant 4$, degree $b_4=4$.
Elimination of u_0 from (4.6) and (4.9) gives

$$\tilde{\varphi}b_4-2\frac{\sigma_1\sigma_3}{t_2t_4}q_4(x^2-\xi^2)=0, \tag{4.10}$$

and then elimination of u_0 from (4.7) and (4.9) leads to $Mq_4(0)=0$.
If $M=0$ we are ready. Therefore suppose $M\neq 0$. Then $q_4(0)=0$. Since $P_2(0)\neq 0$, we obtain $C_2(0)=0$, i.e. by Lemma 4(i) $\xi=0$. Then (4.7) reduces to

$$\tilde{\varphi}u_0=2\frac{\sigma_1\sigma_3}{t_2t_4}x^2u_1+M\delta(0). \tag{4.11}$$

Putting $q_4=x^2q_2$, $b_4=x^2b_2$ we obtain from (4.9) and (4.10)

$$x^2q_2u_0=x^2b_2u_1, \tag{4.12}$$

$$\tilde{\varphi}b_2-\frac{2\sigma_1\sigma_3}{t_2t_4}q_2x^2=0. \tag{4.13}$$

Then elimination of u_1 from (4.11) and (4.12) gives $Mb_2(0)=0$. Since we had assumed $M\neq 0$ we obtain $b_2(0)=0$, i.e.

$$C_2 C_3' - C_3 C_2' = b_4 = x^2 b_2 = \frac{\sigma_1 \sigma_2}{t_2 t_3}\, x^4.$$

It is easy to see that then $C_3 = (\sigma_2/t_3)\, x^3$.

We have found that $M\neq 0$ implies $C_2=(\sigma_1/t_2)\, x^2$ and $C_3=(\sigma_2/t_3)\, x^3$. Then elimination of Du_1 from (4.4) and (4.8) can be done in such a way that one arrives at

$$q_2 u_0 = \frac{\sigma_1 \sigma_2}{t_2 t_3}\, x^2 u_1. \tag{4.14}$$

Relation (4.13) reduces to

$$\tilde{\varphi}\, \frac{\sigma_2}{t_3} - 2\, \frac{\sigma_3}{t_4}\, q_2 = 0. \tag{4.15}$$

Finally (4.11), (4.14) and (4.15) imply $M=0$, a contradiction. This completes the proof of the theorem. ∎

In order to treat the situation where B in (4.3) has two different pairs of zeros $\{-\xi_1, \xi_1\}$ and $\{-\xi_2, \xi_2\}$ we derive a basic relation similar to relation (3.1).

By Proposition 2 and Corollary 2 we have

$$\begin{aligned} &(2n+1)\, \frac{P_{2n+1}}{p_{2n+1}}\, Bu_1 \\ &\quad = (2n+1)\, \frac{P_{2n+1}}{x p_{2n+1}}\, xBu_1 \\ &\quad = (2n+1)\, \frac{P_{2n+1}}{x p_{2n+1}}\, x\varphi u_0 = \varphi D(C_{2n+2} u_1) \\ &\quad = \varphi C_{2n+2}' u_1 + \varphi C_{2n+2}\, Du_1 = (\varphi C_{2n+2}' + C_{2n+2} \pi)\, u_1. \end{aligned}$$

Hence

$$(2n+1)\, \frac{P_{2n+1}}{x p_{2n+1}}\, B = \varphi\, \frac{C_{2n+2}'}{x} + C_{2n+2}\, \frac{\pi}{x}, \qquad n \geqslant 0. \tag{4.16}$$

We have used the fact that P_{2n+1}, C_{2n+2}' and π are odd polynomials.

LEMMA 5. *Let ξ be such that $B(\xi)=0$, $\varphi(\xi)\neq 0$, where B and φ denote the polynomials defined in* (4.3) *and* (4.1). *Then there exists a k independent of n, $k\neq 0$, such that*

$$C_{2n+2}(\xi)+k\,\frac{C'_{2n+2}(\xi)}{\xi}=0 \qquad \text{for all} \quad n\neq 0.$$

Proof. Substitution of ξ in (4.16) gives

$$\varphi(\xi)\,\frac{C'_{2n+2}(\xi)}{\xi}+C_{2n+2}(\xi)\,\frac{\pi(\xi)}{\xi}=0.$$

(If $\xi=0$, then $\pi(\xi)/\xi$ has to be read as $\pi(\xi)/\xi=\lim_{x\to 0}(\pi(x)/x)$; the same should be done with $C'_{2n+2}(\xi)/\xi$.)

The relation with $n=0$ reads

$$\varphi(\xi)\,\frac{C'_2(\xi)}{\xi}+C_2(\xi)\,\frac{\pi(\xi)}{\xi}=0.$$

Since $C'_2(\xi)/\xi=2(\sigma_1/t_2)\neq 0$ and $\varphi(\xi)\neq 0$ it follows $\pi(\xi)/\xi\neq 0$. Then the lemma is satisfied with $k=\varphi(\xi)(\xi/\pi(\xi))\neq 0$. ∎

LEMMA 6. *Suppose that there exist ξ_1, ξ_2, $k_1\neq 0$ and $k_2\neq 0$ such that*

$$C_{2n+2}(\xi_1)+k_1\,\frac{C'_{2n+2}(\xi_1)}{\xi_1}=0 \qquad \text{and} \qquad C_{2n+2}(\xi_2)+k_2\,\frac{C'_{2n+2}(\xi_2)}{\xi_2}=0 \tag{4.17}$$

for all $n\geqslant 0$. Then $\xi_1=\pm\xi_2$ and $k_1=k_2$.

Proof. The polynomials T_{2n} and C_{2n} are even polynomials. Write $T_n^*(x^2)=T_{2n}(x)$ and $C_n^*(x^2)=C_{2n}(x)$, then $\{T_n^*\}$ are orthogonal with respect to the functional u_1^* defined by the moments $\langle u_1^*, x^n\rangle=\langle u_1, x^{2n}\rangle$, $n=0, 1, 2, ...$ The relations (4.17) become

$$C^*_{n+1}(\xi_j^2)+2k_j\,C^{*\prime}_{n+1}(\xi_j^2)=0, \qquad j=1, 2, \quad n\geqslant 0.$$

Proceeding as in the proof of Lemma 2 we obtain $\xi_1^2=\xi_2^2$ and $k_1=k_2$. ∎

LEMMA 7. *Suppose B in* (4.3) *has two different pairs of zeros. Then*

(i) *at least one pair of zeros of B is also a pair of zeros of φ;*

(ii) *if $(x^2-\xi^2)\mid B$ and $(x^2-\xi^2)\mid\varphi$, then $C_2(\xi)\neq 0$ and $(x^2-\xi^2)\mid\pi$.*

Proof. Assertion (i) of the lemma is a direct consequence of Lemma 5 and Lemma 6. If $\{-\xi, \xi\}$ is a common pair of zeros of B and φ, then (4.16) with $n=0$ implies $x^2-\xi^2 \mid C_2(\pi/x)$. Since B has two different pairs of zeros Lemma 4(ii) implies $C_2(\xi)\neq 0$. Hence $x^2-\xi^2 \mid \pi$. ∎

THEOREM 4. *Let $\{u_0, u_1\}$ denote a symmetrically coherent pair of quasi-definite linear functionals. Let B in (4.3) be of the form*

$$B=2\,\frac{\sigma_1\sigma_3}{t_2t_4}\,(x^2-\xi_1^2)(x^2-\xi_2^2) \qquad \textit{with} \quad \xi_1^2\neq\xi_2^2.$$

Then

(i) *u_1 is classical with $D(\tilde{\varphi}u_1)=\psi u_1$ for some polynomials $\tilde{\varphi}$, ψ, degree $\tilde{\varphi}\leqslant 2$, degree $\psi=1$;*

(ii) *there exists $\xi\in\{\xi_1, \xi_2\}$ such that*

$$\tilde{\varphi}u_0=2\,\frac{\sigma_1\sigma_3}{t_2t_4}\,(x^2-\xi^2)\,u_1.$$

Proof. According to Lemma 7(i) we may suppose that $\{-\xi_1, \xi_1\}$ is also a pair of zeros of φ. Then, by Lemma 7(ii), $C_2(\xi_1)\neq 0$ and $\{-\xi_1, \xi_1\}$ is also a pair of zeros of π. Put

$$B=(x^2-\xi_1^2)\,\tilde{B}, \qquad \varphi=(x^2-\xi_1^2)\,\tilde{\varphi}, \qquad \pi=(x^2-\xi_1^2)\,\pi_1.$$

Then (4.16) becomes

$$(2n+1)\,\frac{P_{2n+1}}{p_{2n+1}}\,\tilde{B}=\tilde{\varphi}C'_{2n+2}+C_{2n+2}\pi_1, \qquad n\geqslant 0. \tag{4.18}$$

Moreover, the relations $x\varphi u_0=xBu_1$, $B\,Du_1=\pi u_0$ and $\varphi\,Du_1=\pi u_1$ from Corollary 2 give

$$x^2\tilde{\varphi}u_0=x^2\tilde{B}u_1+M\delta(\xi_1)+M\delta(-\xi_1), \tag{4.19}$$

$$x\tilde{B}\,Du_1=x\pi_1u_0+N\delta(\xi_1)+N\delta(-\xi_1), \tag{4.20}$$

$$x\tilde{\varphi}\,Du_1=x\pi_1u_1+K\delta(\xi_1)+K\delta(-\xi_1), \tag{4.21}$$

where we have used the fact that the functionals applied on polynomials of odd degree have to give zero.

We will show $M=N=K=0$.

It follows from (4.18) and Proposition 2 that

$$(\tilde{\varphi} C'_{2n+2} + C_{2n+2}\pi_1)\, u_0 = (2n+1)\, \frac{P_{2n+1}}{p_{2n+1}}\, \tilde{B} u_0 = \tilde{B}(C'_{2n+2} u_1 + C_{2n+2}\, Du_1).$$

Hence

$$\frac{C'_{2n+2}}{x}\, \{x^2 \tilde{\varphi} u_0 - x^2 \tilde{B} u_1\} = C_{2n+2}\{x\tilde{B}\, Du_1 - x\pi_1 u_0\}, \qquad n \geqslant 0.$$

Then (4.19) and (4.20) imply

$$2\, \frac{C'_{2n+2}(\xi_1)}{\xi_1}\, M = 2C_{2n+2}(\xi_1)\, N, \qquad n \geqslant 0. \tag{4.22}$$

Observe that $C'_2(\xi_1)/\xi_1 = 2(\sigma_1/t_2) \neq 0$, $C_2(\xi_1) \neq 0$; then $M=0$ if and only if $N=0$. Consider the second pair of zeros $\{-\xi_2, \xi_2\}$ of B. There are two possibilities: $\varphi(\xi_2) \neq 0$ and $\varphi(\xi_2) = 0$. If $\varphi(\xi_2) \neq 0$, then Lemma 5 and Lemma 6 imply that relation (4.22) has to be trivial, i.e. $M=N=0$. If $\varphi(\xi_2)=0$, then we can proceed with ξ_2 as with ξ_1 and arrive at a relation for ξ_2 similar to relation (4.22) for ξ_1. Again Lemma 6 implies that at least one of the relations has to be a trivial one, and without loss of generality we may suppose that the relation (4.22) for ξ_1 is trivial. Hence in both cases we obtain $M=N=0$.

In order to prove that $K=0$ we proceed as follows. With (4.4) and (4.18) for $n=0$ we obtain

$$\tilde{\varphi}\, \frac{P_1}{p_1}\, u_0 = \tilde{\varphi}(C'_2 u_1 + C_2\, Du_1) = \left(\frac{P_1}{p_1}\, \tilde{B} - C_2 \pi_1\right) u_1 + \tilde{\varphi} C_2\, Du_1,$$

or

$$\frac{P_1}{xp_1}\, \{x^2 \tilde{\varphi} u_0 - x^2 \tilde{B} u_1\} = C_2\{x\tilde{\varphi}\, Du_1 - x\pi_1 u_1\}.$$

Then (4.19) with $M=0$ and (4.21) imply

$$2KC_2(\xi_1) = 0.$$

Since $C_2(\xi_1) \neq 0$, we have $K=0$.

Now we are able to prove the assertions of the theorem. Relation (4.21) with $K=0$ reads $x\tilde{\varphi}\, Du_1 = x\pi_1 u_1$. Since u_1 is symmetric and $\tilde{\varphi}'$ and π_1 are odd polynomials we have $\langle \tilde{\varphi}\, Du_1, 1\rangle = \langle \pi_1 u_1, 1\rangle = 0$. Then the relation can be reduced to

$$\tilde{\varphi}\, Du_1 = \pi_1 u_1. \tag{4.23}$$

Then $D(\tilde{\varphi} u_1) = \tilde{\varphi}' u_1 + \tilde{\varphi}\, Du_1 = (\tilde{\varphi}' + \pi_1)\, u_1 = \psi u_1$, with degree $\tilde{\varphi} \leqslant 2$, degree $\psi \leqslant 1$. However, ψ is an odd polynomial and $\psi \equiv 0$ is impossible for a quasi-definite functional u_1. Then degree $\psi = 1$ and u_1 is classical. This proves assertion (i) of the theorem.

In the same way (4.19) with $M = 0$ reduces to

$$x\tilde{\varphi} u_0 = x\tilde{B} u_1 \tag{4.24}$$

or

$$\tilde{\varphi} u_0 = \tilde{B} u_1 + L\delta(0). \tag{4.25}$$

Observe that $\tilde{B} = (2\sigma_1 \sigma_3 / t_2 t_4)(x^2 - \xi_2^2)$.

We will prove that $L = 0$ in (4.25), which completes the proof of assertion (ii) of the theorem. Elimination of u_0 from (4.8) and (4.24) gives, using (4.23),

$$C_3' \tilde{\varphi} + C_3 \pi_1 - 2 \frac{P_2}{p_2} \tilde{B} = 0.$$

Then elimination of u_0 from (4.8) and (4.25) gives

$$2 \frac{P_2(0)}{p_2} L = 0.$$

Since $P_2(0) \neq 0$, we obtain $L = 0$. ∎

Theorem 3 and Theorem 4 enables us to give all symmetrically coherent pairs which can be represented by distribution functions. In Theorem 3 and Theorem 4 the ξ may be complex, in the distribution functions below we always assume the ξ to be real.

D. *Hermite Case.* The classical distribution function is $d\Psi(x) = e^{-x^2}\, dx$ on $(-\infty, \infty)$ with $\varphi(x) \equiv 1$. Theorem 3 and Theorem 4 give the symmetrically coherent pairs of distribution functions on $(-\infty, \infty)$

$$\left\{ e^{-x^2}\, dx, \frac{e^{-x^2}}{x^2 + \xi^2}\, dx \right\} \qquad \text{with} \quad \xi \neq 0,$$

$$\{ (x^2 + \xi^2)\, e^{-x^2}\, dx, e^{-x^2}\, dx \}.$$

It is easy to prove that these pairs are indeed symmetrically coherent pairs.

E. *Gegenbauer Case*. The classical distribution function is $d\Psi(x) = (1-x^2)^\alpha\, dx$ on $(-1, 1)$ with $\alpha > -1$; the corresponding functional u satisfies $D(\varphi u) = \psi u$ with $\varphi(x) = 1 - x^2$.

We obtain the following symmetrically coherent pairs of distribution functions with obvious definition of the spectra

$$\left\{(1-x^2)^{\alpha-1}\, dx, \frac{(1-x^2)^\alpha}{x^2+\xi^2}\, dx\right\}, \qquad \alpha > 0, \quad \xi \neq 0,$$

and

$$\left\{(1-x^2)^{\alpha-1}\, dx, \frac{(1-x^2)^\alpha}{\xi^2-x^2}\, dx + M\delta(\xi) + M\delta(-\xi)\right\},$$

with $\alpha > 0$, $|\xi| \geqslant 1$, $M \geqslant 0$.

$$\{(x^2+\xi^2)(1-x^2)^{\alpha-1}\, dx, (1-x^2)^\alpha\, dx\}, \qquad \alpha > 0$$

$$\{(\xi^2-x^2)(1-x^2)^{\alpha-1}\, dx, (1-x^2)^\alpha\, dx\}$$

with $|\xi| > 1$, $\alpha > 0$ and

$$\{dx + M\delta(1) + M\delta(-1), dx\}, \qquad M \geqslant 0.$$

Again one can prove that the mentioned pairs are coherent pairs.

Remark. In [2] the concept of generalized coherent pairs has been introduced. It reads for linear functionals: let u_0 and u_1 denote quasi-definite linear functionals and let $\{P_n\}$ and $\{T_n\}$ denote their MOPS, then $\{u_0, u_1\}$ is called a generalized coherent pair if there exist constants σ_n, τ_n such that

$$T_n = \frac{P'_{n+1}}{n+1} - \sigma_n \frac{P'_n}{n} - \tau_n \frac{P'_{n-1}}{n-1} \qquad \text{for} \quad n \geqslant 2.$$

Let $\alpha > -1$, $\xi_1 < 0$, $\xi_2 < 0$, $M \geqslant 0$, then

$$\left\{x^\alpha e^{-x}\, dx, \frac{1}{x-\xi_2} x^{\alpha+1} e^{-x}\, dx + M\delta(\xi_2)\right\}$$

and

$$\{(x-\xi_1)\, x^\alpha e^{-x}\, dx, x^{\alpha+1} e^{-x}\, dx\}$$

are coherent pairs. From this observation it easily follows that

$$\left\{(x-\xi_1)\, x^{\alpha}e^{-x}\, dx,\ \frac{1}{x-\xi_2}\, x^{\alpha+1}e^{-x}\, dx + M\delta(\xi_2)\right\}$$

is a generalized coherent pair. (Obviously the dx-terms are distribution functions on $[0, \infty)$ and if $M \neq 0$ the last term gives a contribution from ξ_2 outside $(0, \infty)$.) Here none of the distribution functions is a classical one, so the results of this paper cannot be generalized to generalized coherent pairs.

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